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When $D((X))$ and $D\{\{X\}\}$ are Prüfer domains

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ABSTRACT

Let D be an integral domain, X be an indeterminate over D, and $D[\![X]\!]$ be the power series ring over D. For $f \in D[\![X]\!]$, let $c_D(f)$ denote the ideal of D generated by the coefficients of f. Let $N = \{f \in D[\![X]\!] \mid c_D(f) = D\}$, $N_t = \{f \in D[\![X]\!] \mid c_D(f)_t = D\}$, $D((X)) = D[\![X]\!]_N$, and $D\{\{X\}\} = D[\![X]\!]_{N_t}$. We show that D is a Krull domain if and only if $D\{\{X\}\}$ is a Prüfer domain, if and only if $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain for each maximal t-ideal P of D, if and only if $D[\![X]\!]$ is a PvMD in which each t-ideal is divisorial. We also show that D is a Dedekind domain if and only if $D((X))$ is a Prüfer domain, if and only if $D[\![X]\!]_{M[\![X]\!]}$ is a valuation domain for each maximal ideal M of D.

1. Introduction

Let D be an integral domain, X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $D[\![X]\!]$ be the power series ring over D. For any $f \in D[X]$ or $D[\![X]\!]$, we denote by $c_D(f)$ the ideal of D generated by the coefficients of f. Let $D(X) = \{\frac{f}{g} | f, g \in D[X], g \neq 0, \text{ and } c_D(g) = D\}$, $D\{X\} = \{\frac{f}{g} | f, g \in D[X], g \neq 0, \text{ and } c_D(g)_t = D\}$, $D((X)) = \{\frac{f}{g} | f, g \in D[\![X]\!], g \neq 0, \text{ and } c_D(g) = D\}$, and $D\{\{X\}\} = \{\frac{f}{g} | f, g \in D[\![X]\!], g \neq 0, \text{ and } c_D(g)_t = D\}$. (Definitions related to the t-operation will be reviewed in the sequel.)

It is well known that D is a Dedekind domain (resp., Krull domain) if and only if $D(X)$ (resp., $D\{X\}$) is a PID, if and only if $D(X)$ (resp., $D\{X\}$) is a Euclidean domain [1, Theorems 2.4 and 2.6]. As the power series ring analogs, Anderson and Kang proved that D is a Dedekind domain (resp., Krull domain) if and only if $D((X))$ (resp., $D\{\{X\}\}$) is a PID, if and only if $D((X))$ (resp., $D\{\{X\}\}$) is a Euclidean domain [1, Theorems 4.1 and 4.10].

Recall that D is a Prüfer domain (resp., PvMD) if and only if $D(X)$ (resp., $D\{X\}$) is a Prüfer domain, if and only if $D(X)$ (resp., $D\{X\}$) is a Bezout domain [5, Theorem 33.4] (resp., [8, Theorem 3.7]). Also, D is a PvMD in which each t-ideal is divisorial if and only if $D[X]$ is a PvMD in which each t-ideal is divisorial, if and only if D is integrally closed and each nonzero ideal of $D\{X\}$ is divisorial [6, Theorem 2.10 and Corollary 3.6]. In this paper, we study when $D((X))$ and $D\{\{X\}\}$ are Prüfer domains. More precisely, we show that D is a Krull domain if and only if $D\{\{X\}\}$ is a Prüfer domain, if and only if $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain for each maximal t-ideal P of D, if and only if $D[\![X]\!]$ is a PvMD in which each t-ideal is divisorial. We also show that D is a Dedekind domain if and only if $D((X))$ is a Prüfer domain, if and only if $D[\![X]\!]_{M[\![X]\!]}$ is a valuation domain for each maximal ideal M of D.

To facilitate the reading of the paper, we first review definitions related to the t-operation. Let K be the quotient field of D, and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For any $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{J_v | J \text{ is a nonzero finitely generated subideal of } I\}$. An $I \in \mathbf{F}(D)$ is called a *v-ideal* or a *divisorial ideal* (resp., *t-ideal*) if $I_v = I$ (resp., $I_t = I$), while an integral ideal is a *maximal t-ideal* if it is maximal among proper integral t-ideals. An

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easy Zorn's lemma argument shows that each proper integral t-ideal is contained in a maximal t-ideal. It is also easy to see that each maximal t-ideal is a prime ideal. Let t-Max(D) denote the set of maximal t-ideals of D; so $D = \cap_{P\in t\text{-Max}(D)} D_P$ by [5, Exercise 22, page 52]. We say that D has *t-dimension one*, written t-dim$(D) = 1$, if each maximal t-ideal of D has height one. An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. The D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible, while D is a Krull domain if each nonzero ideal of D is t-invertible [9, Theorem 3.6]; so a Krull domain is a PvMD. We know that D is a PvMD (resp., Prüfer domain) if and only if D_P is a valuation domain for all maximal t-ideals (resp., maximal ideals) P of D. Hence D is a Prüfer domain if and only if D is a PvMD and each maximal ideal of D is a t-ideal. It is well known that D is a Dedekind domain if and only if D is a Krull domain and $\dim(D) = 1$, where $\dim(D)$ denotes the (Krull) dimension of D.

2. When $D(\!(X)\!)$ and $D\{\{X\}\}$ are Prüfer domains

Let D be an integral domain, X be an indeterminate over D, and $D[\![X]\!]$ be the power series ring over D, $N = \{f \in D[\![X]\!] | c_D(f) = D\}$, $N_t = \{f \in D[\![X]\!] | c_D(f)_t = D\}$, $D(\!(X)\!) = D[\![X]\!]_N$, and $D\{\{X\}\} = D[\![X]\!]_{N_t}$. Obviously, $N \subseteq N_t$, and thus $D[\![X]\!] \subseteq D(\!(X)\!) \subseteq D\{\{X\}\} = D(\!(X)\!)_{N_t}$.

For an ideal I of D, we let $I[\![X]\!] = \{f \in D[\![X]\!] | c_D(f) \subseteq I\}$, $I(\!(X)\!) = I[\![X]\!]_N$, and $I\{\{X\}\} = I[\![X]\!]_{N_t}$; hence $I[\![X]\!]$ (resp., $I(\!(X)\!)$, $I\{\{X\}\}$) is an ideal of $D[\![X]\!]$ (resp., $D(\!(X)\!)$, $D\{\{X\}\}$). In particular, I is a prime ideal if and only if $I[\![X]\!]$ (resp., $I(\!(X)\!)$, $I\{\{X\}\}$) is a prime ideal.

Lemma 1 (*[2, Theorem 25]*). *Let P be a nonzero prime ideal of D. If $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain, then D_P and $D[\![X]\!]_{P[\![X]\!]}$ are both rank one DVRs. In particular, $ht(P) = \mathrm{ht}(P[\![X]\!]) = 1$.*

Let $X^1(D)$ be the set of height one prime ideals of D. So if t-dim$(D) = 1$, then t-Max$(D) = X^1(D)$. An integral domain D is a Krull domain if and only if (i) D_P is a rank one DVR for each $P \in X^1(D)$, (ii) $D = \cap_{P\in X^1(D)} D_P$, and (iii) $D = \cap_{P\in X^1(D)} D_P$ is a locally finite intersection [5, Section 43].

Lemma 2. *D is a Krull domain if and only if D_P is a rank one DVR and P is the radical of a finitely generated ideal for each maximal t-ideal P of D.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) If $P \in t$-Max(D), then P is of height one, because D_P is a rank one DVR. Hence t-dim$(D) = 1$. Next, recall that if each minimal prime ideal of I is the radical of a finitely generated ideal, then I has a finite number of minimal prime ideals [3, Theorem 2.1]. Hence the intersection $D = \cap_{P\in X^1(D)} D_P$ is locally finite. Thus D is a Krull domain. $\square$

Let $\mathbb{N}_0$ be the set of nonnegative integers. A family $\mathcal{U}$ of subsets of $\mathbb{N}_0$ is called a *nonprincipal ultrafilter* on $\mathbb{N}_0$ if (i) $\emptyset \notin \mathcal{U}$, (ii) if $A, B \subseteq \mathbb{N}_0$, $A \subseteq B$, and $A \in \mathcal{U}$, then $B \in \mathcal{U}$, (iii) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, (iv) if $A \subseteq \mathbb{N}_0$, then either $A \in \mathcal{U}$ or $\mathbb{N}_0 \setminus A \in \mathcal{U}$, and (v) every element in $\mathcal{U}$ is an infinite set. Note that $\mathbb{N}_0 \in \mathcal{U}$ by (ii) and $\{i, i+1, \ldots\} \in \mathcal{U}$ for each nonnegative integer i by (iv) and (v). It is known that there exists a nonprincipal ultrafilter on $\mathbb{N}_0$.

Lemma 3. *Let D be an integral domain of t-dim$(D) = 1$, and let M be a maximal t-ideal of D that is not the radical of a finitely generated ideal. Then there exists a maximal t-ideal P of D with $ht(P[\![X]\!]) \geq 2$.*

Proof. Fix $\rho \in M \setminus \{0\}$, and let $\{M_\alpha\}_{\alpha\in\mathcal{A}}$ be the family of maximal t-ideals, distinct from M, that contain ρ, where $\mathcal{A}$ is a well-ordered index set. Construct a countably infinite subset $\{M_n\}_{n=1}^\infty$ as follows. (This type of construction is due to Loper and Lucas [10]. Or, see [4].)

- Step 1. Let α_1 be the minimum element of $\mathcal{A}$, set $M_1 = M_{\alpha_1}$ and choose an element $\rho_1 \in M$ that is not in M_1.
- Step 2. Let α_2 be the smallest α such that $\rho_1 \in M_{\alpha_2}$ (at least one exists, otherwise M is the only maximal t-ideal that contains both ρ and ρ_1 , and hence $M = \sqrt{(\rho, \rho_1)}$, a contradiction). Next, set $M_2 = M_{\alpha_2}$ and choose $\rho_2 \in (M \cap M_1) \setminus M_2$.
- Step 3. (Recursion step) Let α_n be the smallest $\alpha \in \mathcal{A}$ (necessarily with $\alpha > \alpha_{n-1}$) such that $\rho_i \in M_{\alpha_n}$ for $i = 1, 2, \ldots, n-1$. Set $M_n = M_{\alpha_n}$ and choose $\rho_n \in (M \cap M_1 \cap \cdots \cap M_{n-1}) \setminus M_n$.

This construction builds two countably infinite sets $\{M_n\}_{n=1}^\infty$ and $\{\rho_n\}_{n=1}^\infty$ such that $\rho \in \cap_{n=1}^\infty M_n$ and each ρ_n is in M and M_m for all m except for M_n.

Let $M_0 = M$, and for each $f = \sum_{i=0}^\infty a_n X^n \in D[\![X]\!]$, we define φ_f by

$$\varphi_f(n) = \begin{cases} \min\{i | a_i \notin M_n\} & \text{if } f \notin M_n[\![X]\!] \\ \infty & \text{if } f \in M_n[\![X]\!]. \end{cases}$$

So φ_f is a function from $\mathbb{N}_0$ into $\mathbb{N}_0 \cup \{\infty\}$. For any $f, g \in D[\![X]\!]$, we mean by $g \prec f$ that, for each positive integer k, there is a set $U_k \in \mathcal{U}$ such that $k\varphi_g(n) < \varphi_f(n)$ for all $n \in U_k$. Hence if $g \not\prec f$, then there exists an integer $k \geq 1$ such that, for each $U \in \mathcal{U}$, we have $k\varphi_g(n) \geq \varphi_f(n)$ for some $n \in U$.

It is easy to show that if $f, g \in D[\![X]\!]$, then $\varphi_{fg}(n) = \varphi_f(n) + \varphi_g(n)$ and $\varphi_{f+g}(n) \geq \min\{\varphi_f(n), \varphi_g(n)\}$ for each $n \geq 0$. Using this fact, we can show that for $f \in D[\![X]\!]$, if we set $P(f) = \{g \in D[\![X]\!] | f \prec g\}$, then $P(f) = \emptyset$ or $P(f)$ is a prime ideal of $D[\![X]\!]$ [4, Lemma 2.3].

Set $f = \sum_{i=0}^{\infty} \rho_i X^i$ and $g = \sum_{i=0}^{\infty} \rho_i X^{i2^i}$. Clearly, $\varphi_f(n) = n$ and $\varphi_g(n) = n2^n$ for all integers $n \geq 0$. Let k be a positive integer. Then $\varphi_g(n) - k\varphi_f(n) = n2^n - kn = n(2^n - k)$. So if we choose an integer m with $2^m > k$, then $k\varphi_f(n) < \varphi_g(n)$ for all $n \geq m$. Hence $f \prec g$ because $\{n \in \mathbb{N}_0 | n \geq m\} \in \mathcal{U}$. Thus $g \in P(f)$.

Let $c_D(P(f)) = \sum_{h \in P(f)} c_D(h)$. If $c_D(P(f))_t = D$, then there are some $h_1, \ldots, h_s \in P(f)$ such that $(c_D(h_1) + \cdots + c_D(h_s))_t = D$. Also, if we set $h_i = a_{i0} + a_{i1}X + \cdots$, then there exists an integer $p \geq 1$ such that $(\{a_{ij}\}_{i=1,\ldots,s;j=0,1,\ldots,p})_t = D$. Hence $\min\{\varphi_{h_i}(n)\}_{i=1}^{s} \leq p$ for all $n \geq 0$; so if $n > p$, then $\varphi_f(n) = n > \varphi_{h_i}(n)$ for some i. Hence $f \not\prec h_i$, and thus $h_i \notin P(f)$, a contradiction. So $c_D(P(f))_t \subsetneq D$, and therefore $P(f) \subseteq P[\![X]\!]$ for some $P \in t\text{-Max}(D)$.

Finally, if $P(f) = P[\![X]\!]$, then $g \in P[\![X]\!]$, and hence $\rho_i \in P$ for all $i = 0, 1, 2, \ldots$. Hence $f \in P[\![X]\!] = P(f)$, which is contrary to that $f \not\prec f$. So $P(f) \subsetneq P[\![X]\!]$, and therefore $\text{ht}(P[\![X]\!]) \geq 2$. □

An integral domain D is a *Mori domain* if D satisfies the ascending chain condition on integral v-ideals; equivalently, each v-ideal is of finite type, i.e., if $I \in \mathbf{F}(D)$ is a v-ideal, then there exists a finitely generated ideal J of D such that $I = J_v$. Clearly, if I is a nonzero finitely generated ideal, then $I_v = I_t$. Thus each t-ideal of a Mori domain is a v-ideal. It is well known that D is a Krull domain if and only if D is a Mori domain and a PvMD [9, Theorem 3.2]; hence a Krull domain is a PvMD in which each t-ideal is a v-ideal.

Lemma 4. *If D is a Mori domain, then $Max(D\{\{X\}\}) = \{P\{\{X\}\} | P \in t\text{-}Max(D)\}$.*

Proof. Let Q be a prime ideal of $D[\![X]\!]$ such that $Q \not\subseteq P[\![X]\!]$ for all $P \in t\text{-Max}(D)$. Choose $0 \neq f \in Q$. Since D is a Mori domain, there are finitely many maximal t-ideals of D that contain $c_D(f)$ [9, Theorem 2.1]; so we can choose another $g \in Q$ so that $(c_D(f) + c_D(g))_t = D$. Let $f = \sum_{i=0}^{\infty} a_i X^i$; then $c_D(f)_t = (a_0, a_1, \ldots, a_m)_t$ for some m because D is a Mori domain. So if we set $h = f + X^{m+1}g$, then $h \in Q \cap N_t$. (For if $c_D(h) \subseteq P$ for some maximal t-ideal P of D, then $c_D(f) \subseteq P$, and hence $c_D(g) \subseteq P$, a contradiction.) This shows that if Q_0 is a prime ideal of $D[\![X]\!]$ with $Q_0 \subseteq \cup_{P \in t\text{-Max}(D)} P[\![X]\!]$, then $Q_0 \subseteq P[\![X]\!]$ for some $P \in t\text{-Max}(D)$. Thus the proof is completed by [5, Proposition 4.8]. □

We are now ready to prove the main result of this paper, which also gives a new characterization of Krull domains.

Theorem 5. *The following statements are equivalent for an integral domain D.*

(1) *D is a Krull domain.*
(2) *$D[\![X]\!]$ is a Krull domain.*
(3) *$D\{\{X\}\}$ is a Prüfer domain.*
(4) *$D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain for each maximal t-ideal P of D.*
(5) *$D[\![X]\!]$ is a PvMD in which each t-ideal is divisorial.*

Proof. (1) ⇔ (2) [5, Corollaries 44.10 and 44.11].

(1) ⇒ (3) Note that $D[\![X]\!]$ is a Krull domain by the "(1) ⇔ (2)" above; so $P[\![X]\!]$ is a prime t-ideal of $D[\![X]\!]$ for all maximal t-ideals P of D, and hence $D\{\{X\}\}_{P\{\{X\}\}} = D[\![X]\!]_{P[\![X]\!]}$ is a rank one DVR. Note also that $\text{Max}(D\{\{X\}\}) = \{P\{\{X\}\} | P \in t\text{-Max}(D)\}$ by Lemma 4, because a Krull domain is a Mori domain. Thus $D\{\{X\}\}$ is a Prüfer domain.

(3) ⇒ (4) If P is a maximal t-ideal of D, then $P[\![X]\!] \cap N_t = \emptyset$. Hence $P\{\{X\}\} = P[\![X]\!]_{N_t}$ is a proper prime ideal of $D\{\{X\}\}$, and thus $D[\![X]\!]_{P[\![X]\!]} = D\{\{X\}\}_{P\{\{X\}\}}$ is a valuation domain.

(4) ⇒ (1) Let P be a maximal t-ideal of D. Then $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain, and thus D_P is a rank one DVR and $\text{ht}(P[\![X]\!]) = 1$ by Lemma 1. Hence $t\text{-dim}(D) = 1$ and, by Lemma 3, each maximal t-ideal of D is the radical ideal of a finitely generated ideal. Thus D is a Krull domain by Lemma 2.

(2) ⇒ (5) This follows because a Krull domain is a PvMD in which each t-ideal is divisorial.

(5) ⇒ (4) Let P be a maximal t-ideal of D. Then $D[\![X]\!] \supsetneq (PD[\![X]\!])_t = (PD[\![X]\!])_v = (P[\![X]\!])_v = P_v[\![X]\!]$ by (5) and [1, Theorem 3.4]. Hence $P_v \subsetneq D$, and since P is a maximal t-ideal, $P_v = P$. So $(P[\![X]\!])_v = P[\![X]\!]$, and thus $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain. □

Corollary 6. *The following statements are equivalent for an integral domain D.*

(1) *D is a Krull domain.*
(2) *$D(\!(X)\!)$ is a Krull domain.*
(3) *$D\{\{X\}\}$ is a Krull domain.*
(4) *$D\{\{X\}\}$ is a Dedekind domain.*
(5) *$D\{\{X\}\}$ is a PID.*
(6) *$D\{\{X\}\}$ is a Euclidean domain.*
(7) *$D\{\{X\}\}$ is a Bezout domain.*
(8) *$D[\![X]\!]$ is a PvMD and each maximal t-ideal of D is divisorial.*
(9) *$D[\![X]\!]$ is a PvMD and $P[\![X]\!]$ is a t-ideal for each maximal t-ideal P of D.*
(10) *$D[\![X]\!]$ is integrally closed and each nonzero ideal of $D\{\{X\}\}$ is divisorial.*

Proof. (1) $\Rightarrow$ (2) This follows from [5, Corollary 43.6], because $D[\![X]\!]$ is a Krull domain by Theorem 5 and $D(\!(X)\!) = D[\![X]\!]_N$. (2) $\Rightarrow$ (3) Note that $N \subseteq N_t$, and so $D\{\!\{X\}\!\} = D(\!(X)\!)_{N_t}$. Thus $D\{\!\{X\}\!\}$ is a Krull domain [5, Corollary 43.6]. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) [1, Theorem 4.1]. (5) $\Rightarrow$ (4) $\Rightarrow$ (3) Clear. (6) $\Rightarrow$ (7) Clear. (7) $\Rightarrow$ (1) Since a Bezout domain is a Prüfer domain, D is a Krull domain by Theorem 5. (1) $\Rightarrow$ (8) By Theorem 5, $D[\![X]\!]$ is a PvMD in which each t-ideal is divisorial. Thus the proof of the "(5) $\Rightarrow$ (4)" of Theorem 5 shows that each maximal t-ideal of D is divisorial. (8) $\Rightarrow$ (9) This follows because $(P[\![X]\!])_v = P_v[\![X]\!]$ [1, Theorem 3.4]. (9) $\Rightarrow$ (1) If P is a maximal t-ideal of D, then $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain by (9). Thus D is a Krull domain by Theorem 5. (5) $\Rightarrow$ (10) Note that $D[\![X]\!]$ is a PvMD by (5) $\Rightarrow$ (1) $\Rightarrow$ (8) above; so $D[\![X]\!]$ is integrally closed. Thus the result follows because each nonzero principal ideal is divisorial. (10) $\Rightarrow$ (1) Recall that $D\{\!\{X\}\!\} = D[\![X]\!]_{N_t}$; so $D\{\!\{X\}\!\}$ is integrally closed. Hence $D\{\!\{X\}\!\}$ is a Prüfer domain [7, Theorem 5.1], and thus D is a Krull domain by Theorem 5. □

The next result is a Dedekind domain analog of Theorem 5 and Corollary 6. This also gives a new characterization of Dedekind domains.

Corollary 7. *The following statements are equivalent for an integral domain D.*

(1) *D is a Dedekind domain.*
(2) *$D(\!(X)\!)$ is a Dedekind domain*
(3) *$D(\!(X)\!)$ is a PID.*
(4) *$D(\!(X)\!)$ is a Euclidean domain.*
(5) *$D(\!(X)\!)$ is a Bezout domain.*
(6) *$D(\!(X)\!)$ is a Prüfer domain.*
(7) *$D[\![X]\!]_{M[\![X]\!]}$ is a valuation domain for each maximal ideal M of D.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) [1, Theorem 4.10].

(3) $\Rightarrow$ (5) $\Rightarrow$ (6) Clear. (6) $\Rightarrow$ (7) If M is a maximal ideal of D, then $M[\![X]\!] \cap N = \emptyset$. Hence $M(\!(X)\!) = M[\![X]\!]_N$ is a proper prime ideal of $D(\!(X)\!)$, and thus $D[\![X]\!]_{P[\![X]\!]} = D(\!(X)\!)_{M(\!(X)\!)}$ is a valuation domain. (7) $\Rightarrow$ (1) If M is a maximal ideal of D, then $\mathrm{ht}(M[\![X]\!]) = 1$ and D_M is a rank one DVR by (7) and Lemma 1. Hence $\dim(D) = 1$ and, by Lemma 3, each maximal ideal of D is the radical of a finitely generated ideal. By Lemma 2, D is a Krull domain of $\dim(D) = 1$, and thus D is a Dedekind domain. □

Remark 8. (1) In [4, Remark 2.13], the authors noted that if t-$\dim(D) = 1$ and if D has a maximal t-ideal that is not the radical of a finitely generated ideal, then $\dim(D[\![X]\!]) \geq 2^{\aleph_1}$ under the continuum hypothesis that $2^{\aleph_0} = \aleph_1$.

(2) Let Φ be the set of all functions $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\lim_{n\to\infty} \phi(n) = \infty$. For any $\phi, \varphi \in \Phi$, we mean by $\phi \prec \varphi$ that for each positive integer k, there is a set $U_k \in \mathcal{U}$ such that $k\phi(n) < \varphi(n)$ for all $n \in U_k$. Also, we mean by $\phi \sim \varphi$ that there are a positive integer k and a set $U \in \mathcal{U}$ such that $k\phi(n) \geq \varphi(n)$ and $k\varphi(n) \geq \phi(n)$ for all $n \in U$. Then $\sim$ is an equivalence relation on Φ. Let $[\Phi] = \Phi/\sim$ be the set of equivalence classes of elements in Φ. For each $\varphi \in \Phi$, put $[\varphi] = \{\psi \in \Phi | \varphi \sim \psi\}$; hence $[\Phi] = \{[\varphi] | \varphi \in \Phi\}$. For $\varphi, \psi \in \Phi$, we define $[\varphi] \prec [\psi]$ if $\varphi \prec \psi$. Then $([\Phi], \prec)$ is a totally ordered set [4, Proposition 1.2] and the cardinality of $[\Phi]$ is at least $\geq 2^{\aleph_1}$ [4, Theorems 1.11 and 1.12]. Let Φ_0 be a set of representatives of $[\Phi]$ so that $[\Phi] = \{[\varphi] | \varphi \in \Phi_0\}$.

Let the notations be as in the proof of Lemma 3. Recall that if $\phi \in \Phi_0$, then there exists an $f \in M[\![X]\!]$ so that $\varphi_f = \phi$ [4, Lemma 2.4]. For each $\phi \in \Phi_0$, let $f_\phi \in M[\![X]\!]$ such that $\varphi_{f_\phi} = \phi$. Then $\varphi \prec \phi \Leftrightarrow f_\varphi \prec f_\phi$ for each $\phi, \varphi \in \Phi$, and so if we set $\Omega = \{f_\phi \in M[\![X]\!] | \phi \in \Phi_0\}$ and $\mathfrak{P} = \{P(f_\phi) | f_\phi \in \Omega\}$, then $\mathfrak{P}^* = \{\cup_{P\in\mathcal{A}} P | \emptyset \neq \mathcal{A} \subseteq \mathfrak{P}\}$ is a totally ordered set of prime ideals with cardinality $\geq 2^{\aleph_1}$ [4, Proof of Theorem 2.6]. Note that $\lim_{n\to\infty} f_\phi(n) = \infty$ for all $f_\phi \in \Omega$; so the proof of Lemma 3 also shows that $P(f_\phi) \subseteq P[\![X]\!]$ for some maximal t-ideal P of D. Thus, in Lemma 3, $\mathrm{ht}(P[\![X]\!]) \geq 2^{\aleph_1}$.

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